

# A Note on the Cohomology of Finite-Dimensional Cocommutative Hopf Algebras

John H. Palmieri\*

*Department of Mathematics, Room 2-388, Massachusetts Institute of Technology,*

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In the context of finite-dimensional cocommutative Hopf algebras, we prove versions of various group cohomology results: the Quillen–Venkov theorem on detecting nilpotence in group cohomology, Chouinard’s theorem on determining whether a  $kG$ -module is projective by restricting to elementary abelian  $p$ -subgroups of  $G$ , and Quillen’s theorem which identifies the cohomology of  $G$ , “modulo nilpotent elements.” This last result is only proved for graded connected Hopf algebras. © 1997 Academic Press

## 1. INTRODUCTION

When one studies the mod  $p$  cohomology of finite groups, one sees that elementary abelian  $p$ -subgroups play an important role. For example, one can detect nilpotence in group cohomology by restricting to elementary abelian subgroups, one can detect projectivity of modules, and up to  $F$ -isomorphism one can describe group cohomology. In this paper we try to generalize all of this to finite-dimensional cocommutative Hopf algebras; Wilkerson laid the foundations for this work in his paper [15]. Since the setting is more general, the analogues of elementary abelian subgroups are not as pleasant.

Unless otherwise indicated, every Hopf algebra in this paper will be finite-dimensional, graded, cocommutative, and defined over a field  $k$  of characteristic  $p > 0$ . We require cocommutativity for two reasons: so that the cohomology is a commutative  $k$ -algebra with an action of the Steenrod algebra [8], and so that the Hopf algebra is free as a left module over every sub-Hopf algebra [13].

\*E-mail address: [palmieri@math.mit.edu](mailto:palmieri@math.mit.edu).

DEFINITION 1.1. A Hopf algebra  $C$  over  $k$  is *elementary* if it is bicommutative and has  $x^p = 0$  for all  $x \in IC$ , the augmentation ideal of  $C$  (we place no condition on the comultiplication, other than cocommutativity). We will use  $C(x)$  to denote a monogenic elementary Hopf algebra, generated by  $x$ ; i.e.,  $C(x)$  is isomorphic as an algebra to  $k[x]/(x^n)$ , where  $n$  is  $p$  or  $2$ . For a Hopf algebra  $E$ , we call a nonzero element  $v \in \text{Ext}_E^2(k, k)$  a *Serre element* if there is a Hopf algebra extension

$$E' \hookrightarrow E \twoheadrightarrow C(x)$$

so that under the induced map in  $\text{Ext}$ ,  $v$  is the image of a nonzero element of  $\text{Ext}_{C(x)}^2(k, k)$ . A Hopf algebra  $E$  is *quasi-elementary* if no product of Serre elements is nilpotent. (See the Appendix for an alternate description of Serre elements and quasi-elementary Hopf algebras.)

In the  $p$ -group setting, “elementary” and “quasi-elementary” are the same notion by a theorem of Serre [14]—they both correspond to the group algebras of elementary abelian  $p$ -groups—but they differ for more general Hopf algebras. Wilkerson [15] has shown that for every  $p$ , there is a (graded connected) nonelementary quasi-elementary Hopf algebra  $E$  over a field of characteristic  $p$ . For cohomological purposes, quasi-elementary Hopf algebras carry more information; our main theorems illustrate this. In other words, quasi-elementary Hopf algebras seem to be the correct generalization of elementary abelian  $p$ -groups. As such, it would be nice to have alternate characterizations of them; see the Appendix for a few examples.

We move on to the statements of the main results. These are analogous to group cohomology theorems of Carlson [3], Chouinard [4], and Quillen [12]. The first is a generalization of a theorem of Wilkerson [15]; see also [10] (and we have patterned the statement on the nilpotence theorem of [5]). For all of these,  $B$  is a finite-dimensional graded cocommutative Hopf algebra over  $k$ ,  $\text{char}(k) = p$ .

For a class  $z \in \text{Ext}_B^*(L, M)$  and a sub-Hopf algebra  $E$  of  $B$ , we write  $\text{res}_{B, E}(z)$  for its restriction to  $\text{Ext}_E^*(L, M)$ . Also, we say that  $z$  is *tensor-nilpotent* if the element  $z^{\otimes n} \in \text{Ext}_B^*(L^{\otimes n}, M^{\otimes n})$  is zero for  $n$  sufficiently large.

THEOREM 1.2. (a) Let  $\Gamma$  be an associative  $B$ -algebra with unit  $\eta: k \rightarrow \Gamma$ ; fix  $z \in \text{Ext}_B^*(k, \Gamma)$ . Then  $z$  is nilpotent if and only if  $\text{res}_{B, E}(z)$  is nilpotent for every quasi-elementary sub-Hopf algebra  $E$  of  $B$ .

(b) Let  $M$  be a  $B$ -module; fix  $z \in \text{Ext}_B^*(M, M)$ . Then  $z$  is nilpotent if and only if  $\text{res}_{B, E}(z)$  is nilpotent for every quasi-elementary sub-Hopf algebra  $E$  of  $B$ .

(c) Let  $L$  and  $M$  be  $B$ -modules; fix  $z \in \text{Ext}_B^*(L, M)$ . Then  $z$  is tensor-nilpotent if and only if  $\text{res}_{B, E}(z)$  is tensor-nilpotent for every quasi-elementary sub-Hopf algebra  $E$  of  $B$ .

**THEOREM 1.3.** *Let  $M$  be a  $B$ -module. Then  $M$  is projective if and only if  $M$  restricted to  $E$  is projective for every quasi-elementary sub-Hopf algebra  $E$  of  $B$ .*

**THEOREM 1.4.** *Suppose that  $B$  is connected. Let  $\mathcal{A}_B$  be the category with objects the quasi-elementary sub-Hopf algebras of  $B$  and their sub-Hopf algebras, and morphisms given by inclusions. Then the natural map*

$$q_B: \text{Ext}_B^*(k, k) \rightarrow \varprojlim_{E \in \mathcal{A}_B} \text{Ext}_E^*(k, k)$$

*is an  $F$ -isomorphism: the range is finitely generated as a module over  $\text{Ext}_B^*(k, k)$ , every element in the kernel of  $q_B$  is nilpotent, and for every  $z$  in the range, there is an integer  $n$  so that  $z^{p^n}$  is in the image of  $q_B$ .*

We make a few comments about Theorem 1.4. First, it is not clear that in general a sub-Hopf algebra of a quasi-elementary Hopf algebra is again quasi-elementary; hence we have to include these sub-Hopf algebras explicitly in  $\mathcal{A}_B$ . Next, as in the group setting, this theorem has a useful reformulation in terms of varieties, and can also be extended to a similar result about  $\text{Ext}_B^*(M, M)$  (see [9]). Also, note that in the corresponding statement for a group  $G$ , the objects in the category  $\mathcal{A}_G$  are elementary abelian  $p$ -subgroups  $E$  of  $G$ , and the morphisms are given by inclusions and conjugations (and composites of these). In the graded connected Hopf algebra case, we do not need conjugations: in the group case, given  $E$  an elementary abelian subgroup of  $G$ , the conjugations are needed to reflect the action on  $H^*(E)$  by the normalizer of  $E$ —the image of the restriction map from  $H^*(G)$  is contained in the invariants. By [6, Theorem 4.12], though, in the Hopf algebra case all of  $H^*(E)$  is in the image of the restriction map, up to nilpotence.

It would be nice to remove the connectedness condition in Theorem 1.4. Our proof will certainly not generalize, because we deduce Theorem 1.4 from the work of Hopkins and Smith, and they make heavy use of connectedness. In any case, we have the following conjecture.

**CONJECTURE 1.5.** *Let  $B$  be a finite-dimensional cocommutative Hopf algebra; let  $\mathcal{A}_B$  be the category with objects the quasi-elementary sub-Hopf algebras of  $B$  and their sub-Hopf algebras, and morphisms given by inclusions and conjugations by elements of  $B$ . Then the natural map*

$$q_B: \text{Ext}_B^*(k, k) \rightarrow \varprojlim_{E \in \mathcal{A}_B} \text{Ext}_E^*(k, k)$$

*is an  $F$ -isomorphism.*

The structure of the paper is as follows: in Section 2 we explain our notation and review a few facts about Hopf algebras; in Section 3 we prove Theorems 1.2 and 1.3; we prove Theorem 1.4 in Section 4; and we give some examples of quasi-elementary Hopf algebras in the appendix.

## 2. NOTATION AND CONVENTIONS

Throughout the paper all Hopf algebras are graded, cocommutative, and defined over a field  $k$  of characteristic  $p > 0$  (cocommutative is in the usual graded sense).  $E[x]$  (respectively,  $D[x]$ ) will denote any such Hopf algebra which is isomorphic, as an algebra, to  $k[x]/(x^2)$  (resp.,  $k[x]/(x^p)$ ). (Note that if  $p$  is odd, then  $E[x]$  has a Hopf algebra structure only if  $x$  is in an odd grading.) We put no restriction on the coalgebra structure, aside from cocommutativity. We will also use  $C(x)$  to denote either  $E[x]$  or  $D[x]$ , so  $C(x)$  is an arbitrary monogenic elementary Hopf algebra.

If  $B$  is a Hopf algebra, we use  $H^*(B)$  to denote  $\text{Ext}_B^*(k, k)$ , as usual. The augmentation ideal of  $B$  is written  $IB$ . We write  $A \leq B$  to mean that  $A$  is a sub-Hopf algebra of  $B$ ; similarly,  $A \triangleleft B$  means that  $A$  is a normal sub-Hopf algebra of  $B$  (so  $IA \cdot B = B \cdot IA$ ). If  $A \leq B$ , we write  $\text{res}_{B,A}$  for the restriction map

$$\text{res}_{B,A}: \text{Ext}_B^*(-, -) \rightarrow \text{Ext}_A^*(-, -).$$

If  $A \triangleleft B$ , then  $B//A$  is the quotient Hopf algebra  $B \otimes_A k = B/(B \cdot IA)$ . If  $\chi$  denotes the anti-automorphism of  $B$ , then we can define *conjugation* by  $b \in B$  to be the map

$$\begin{aligned} B &\xrightarrow{c_b} B, \\ a &\mapsto \sum_i b_i' a \chi(b_i''), \end{aligned}$$

where  $b$  has coproduct  $\sum_i b_i' \otimes b_i''$ . Then a sub-Hopf algebra  $A$  of  $B$  is normal if and only if  $c_b(a) \in A$  for all  $a \in A$ ,  $b \in B$  (see [11]). There are a number of other similarities between normal sub-Hopf algebras and normal subgroups; we will need the following: if  $A, A' \triangleleft B$ , then  $AA' \leq B$ ; furthermore, we have  $A \cap A' \triangleleft A'$ ,  $A \triangleleft AA'$ , and there is an isomorphism

$$A'/(A \cap A') \cong (AA')//A.$$

(The proof is a straightforward verification.)

Note that for any cocommutative Hopf algebra  $B$ , the cohomology of  $B$  has an action of the Steenrod algebra (see [8, Sect. 11]); we use May's notation for indexing of Steenrod operations on Hopf algebra cohomology.

## 3. PROOFS OF THEOREMS 1.2 AND 1.3

We prove Theorems 1.2 and 1.3 by imitating proofs for the group theory analogues. We start with the following definition; this replaces the product of the Bocksteins of the one-dimensional classes in group cohomology.

DEFINITION 3.1. Suppose  $B$  is not quasi-elementary, so that there is some product of Serre elements which is nilpotent. Define the *fundamental class* of  $B$ ,  $u_B$ , to be any such product. (This is far from unique, of course.)

The following is our key lemma, analogous to [15, 5.2] and [2, Vol. II, 5.2.1].

LEMMA 3.2. Suppose  $v$  is a Serre element, corresponding to the extension

$$A \rightarrow B \rightarrow C(x).$$

Let  $\Gamma$  be an associative  $B$ -algebra with unit  $\eta: k \rightarrow \Gamma$ , and fix  $z \in \text{Ext}_B^*(k, \Gamma)$ . If  $\text{res}_{B,A}(z) = 0$ , then  $z^2 = \eta_*(v)z'$  for some  $z'$ .

*Proof.* This is proved just as in Benson [2]; he attributes the proof to Kroll [7]. We prove this for the case  $C(x) = E[x]$  and leave the other case to the reader. Note that in the  $E[x]$  case, the Serre element  $v$  is the square of some class  $w \in \text{Ext}_B^1(k, k)$ . (For the other case,  $v = \beta\tilde{\mathcal{P}}^0(w)$  for some  $w \in \text{Ext}_B^1(k, k)$ .)

There is a short exact sequence of  $E[x]$ -modules, and hence of  $B$ -modules:

$$0 \rightarrow k \rightarrow E[x] \xrightarrow{q} k \rightarrow 0. \quad (1)$$

This module extension corresponds to  $w \in \text{Ext}_B^1(k, k)$ . Note that the composite

$$\text{Ext}_B^*(k, \Gamma) \xrightarrow{q^*} \text{Ext}_B^*(E[x], \Gamma) \cong \text{Ext}_A^*(k, \Gamma)$$

is the restriction map (see [2, Vol. I, Sect. 2.8]). Consider the long exact sequence in  $\text{Ext}_B^*(-, \Gamma)$  associated to the extension (1): since  $q^*(z) = 0$  then  $z$  is in the image of the boundary homomorphism; but the boundary homomorphism is multiplication by  $\eta_*(w)$ . Since this element is central in  $\text{Ext}_B^*(k, \Gamma)$ , then  $z = \eta_*(w)y$  for some  $y$ ; so the result follows. ■

*Proof of Theorem 1.2.* First we prove part (a); the other parts will follow from this.

This is proved by induction on the sub-Hopf algebras of  $B$  using Lemma 3.2, just as in [15]. If  $A \leq B$  and  $\dim_k(A) = 1$ , then  $A = k$  and the theorem is trivially true for  $z \in \text{Ext}_A^*(k, \Gamma)$ . This starts the induction.

Suppose the theorem holds for all proper sub-Hopf algebras  $A$  of  $B$ , and suppose  $z \in \text{Ext}_B^*(k, \Gamma)$  is nilpotent upon restriction to every quasi-elementary sub-Hopf algebra of  $B$ . We may assume that  $B$  is not quasi-elementary, so that  $B$  has a fundamental class  $u_B$ . For each Serre element  $v$  which is a factor in  $u_B$ , we have an extension

$$A \rightarrow B \rightarrow C(x).$$

By induction, we know that  $\text{res}_{B,A}(z)$  is nilpotent, and so we may replace  $z$  with a large enough power so that  $\text{res}_{B,A}(z) = 0$ . Hence  $z^2 = \eta_*(v)z''$  for some  $z''$ . Since this is true for every factor of  $u_B$ , we see that  $z^r = \eta_*(u_B)\zeta$  for some integer  $r$  and some  $\zeta \in \text{Ext}_B^*(k, \Gamma)$ . Since  $\eta_*(u_B)$  is nilpotent and central in  $\text{Ext}_B^*(k, \Gamma)$ , then  $z$  is nilpotent.

Part (b) follows from (a) via the isomorphism

$$\text{Ext}_B^*(M, M) \cong \text{Ext}_B^*(k, \text{Hom}_k(M, M))$$

and the observation that  $\text{Hom}_k(M, M)$  is an associative  $B$ -algebra with unit.

Part (c) also follows easily: we have  $\text{Ext}_B^*(L, M) \cong \text{Ext}_B^*(k, \text{Hom}_k(L, M))$ , so let  $\Gamma$  be the tensor-algebra of the  $B$ -module  $\text{Hom}_k(L, M)$ :

$$\Gamma = \bigoplus_{n \geq 0} (\text{Hom}_k(L, M))^{\otimes n}.$$

Then  $z \in \text{Ext}_B^*(L, M)$  is tensor-nilpotent if and only if the corresponding element  $\tilde{z} \in \text{Ext}_B^*(k, \Gamma)$  is nilpotent. ■

*Remark 3.3.* The mod 2 Steenrod algebra  $\mathcal{A}$  is a union of finite-dimensional sub-Hopf algebras, and one can prove Theorem 1.2 for  $\mathcal{A}$  (see [10]). (Note that there are no Serre elements in  $\text{Ext}_{\mathcal{A}}^*(\mathbf{F}_2, \mathbf{F}_2)$ ; so in this context (graded, connected, possibly infinite-dimensional Hopf algebras), we add to the definition of quasi-elementary that there exist Serre elements in  $\text{Ext}_E^2(k, k)$ .) It may be worthwhile to examine detection of nilpotence in  $\text{Ext}_B$ , for any cocommutative Hopf algebra  $B$  which is the union of finite-dimensional sub-Hopf algebras.

We move on to a discussion of Theorem 1.3.

**LEMMA 3.4.** *Let  $B$ ,  $v$ , and  $A$  be as in Lemma 3.2. Let  $M$  and  $N$  be  $B$ -modules. Define the function*

$$\mu: \text{Ext}_B^*(M, N) \rightarrow \text{Ext}_B^*(M, N)$$

*to be composition with  $v \otimes 1_M$ . If the restriction of  $M$  to  $A$  is projective, then  $\mu$  is an isomorphism in positive degrees.*

*Proof.* This is proved as in [4]; namely, we observe that the spectral sequence

$$\text{Ext}_{C(x)}^*(k, \text{Ext}_A^*(M, N)) \Rightarrow \text{Ext}_B^*(M, N)$$

associated to the extension

$$A \rightarrow B \rightarrow C(x)$$

collapses. ■

*Proof of Theorem 1.3.* First, if  $M$  is projective over  $B$ , then by a result of Radford [13],  $M$  is projective over every  $E \leq B$ . The converse is proved as in [4]: we may assume that  $B$  is not quasi-elementary; let  $u_B$  be a fundamental class for  $B$ . Fix an arbitrary  $B$ -module  $N$ . By induction on the factors of  $u_B$  and Lemma 3.4, multiplication by  $u_B \otimes 1_M$  is an isomorphism on  $\text{Ext}_B^*(M, N)$  for  $*$   $>$   $0$ . But  $u_B$  is nilpotent; hence  $\text{Ext}_B^*(M, N) = 0$  for  $*$   $>$   $0$ . Since this holds for all  $B$ -modules  $N$ , then  $\text{Hom}_B(M, -)$  is exact, and so  $M$  is projective over  $B$ . ■

*Remark 3.5.* For the sub-Hopf algebras of the mod 2 Steenrod algebra  $A$ , the notions of elementary and quasi-elementary are the same (see [15]). So Theorem 1.3 provides an almost immediate proof of a theorem of Adams and Margolis [1], that a bounded below  $A$ -module  $M$  is free if and only if  $H(M, P_t^s) = 0$  for all  $P_t^s \in A$  with  $s < t$ . Indeed, their theorem is basically a special case of Theorem 1.3, combined with [15, Theorem 6.4]. Here is a related conjecture; this would provide a replacement for the Adams–Margolis theorem when working with nonbounded below modules.

**CONJECTURE 3.6.** *Theorem 1.3 holds for arbitrary sub-Hopf algebras of the mod 2 Steenrod algebra. In particular, it holds for the full Steenrod algebra.*

#### 4. PROOF OF THEOREM 1.4

Let  $B$  be a graded connected cocommutative Hopf algebra over a field  $k$  of characteristic  $p$ . Our goal here is to prove Theorem 1.4, the analogue of the Quillen stratification theorem. It follows from Wilkerson’s work on finite generation in [15], Theorem 1.2 for  $M = k$ , and the next result.

**THEOREM 4.1.** *Let  $\mathcal{C}$  be a poset of sub-Hopf algebras of  $B$  which is closed under inclusions—if  $A \in \mathcal{C}$  and  $A' \leq A$ , then  $A' \in \mathcal{C}$ . Then the natural map*

$$q: \text{Ext}_B^*(k, k) \rightarrow \varprojlim_{A \in \mathcal{C}} \text{Ext}_A^*(k, k)$$

*is an  $F$ -surjection: for every  $z$  in the range, there is an integer  $n$  so that  $z^{p^n}$  is in the image of  $q$ .*

This in turn essentially follows from a theorem of Hopkins and Smith [6, Theorem 4.12]: if  $A$  is a sub-Hopf algebra of  $B$  and  $z \in H^*(A)$ , then for some  $n$ ,  $z^{p^n}$  is in the image of  $\text{res}_{B, A}$ .

*Proof.* The goal here is to show that if we have a compatible family of elements  $\{z_A \in H^*(A) : A \in \mathcal{C}\}$ , then there is an element  $y \in H^*(B)$  and an integer  $n$  so that  $\text{res}_{B, A}(y) = (z_A)^{p^n}$  for each  $A$ .

By induction on the sub-Hopf algebras of  $B$ , it suffices to consider the case where  $\mathcal{C}$  is the collection of all proper sub-Hopf algebras of  $B$ : let  $B'$  be a minimal sub-Hopf algebra of  $B$  not in  $\mathcal{C}$ ; then  $\mathcal{C}$  contains all of the proper sub-Hopf algebras of  $B'$ , so there is an element  $z$  of  $H^*(B')$  compatible with powers of the other  $z_A$ 's, and hence we can add  $B'$  to  $\mathcal{C}$  (after replacing the  $z_A$ 's with large enough powers—note that each  $B'$  has finitely many sub-Hopf algebras).

We have the following lemmas.

**LEMMA 4.2.** *If  $A$  is a maximal sub-Hopf algebra of  $B$ , then  $A$  is normal and  $B//A \cong C[u]$ . Slightly more generally, if  $A \leq B$  and  $A' \leq B$  are maximal with  $A \neq A'$ , then*

$$A'/(A \cap A') \cong B//A \cong C[u].$$

*Proof.* For the first statement, see [6, Lemma A.11]. To identify the quotient Hopf algebra in the second statement, note that  $AA' = B$ —since  $A, A' \triangleleft B$ , then  $AA'$  is a sub-Hopf algebra of  $B$ , and it properly contains the maximal sub-Hopf algebra  $A$ . Now, use the isomorphism given in Section 2:

$$A'/(A \cap A') \cong AA'/A. \quad \blacksquare$$

Let  $A_1, A_2, \dots, A_n$  be the (distinct) maximal elements of  $\mathcal{C}$ . Then for each  $i$  we have a Hopf algebra extension

$$A_i \rightarrow B \rightarrow C[u_i].$$

Let  $v_i \in \text{Ext}_B^2(k, k)$  be the corresponding Serre element.

**LEMMA 4.3.** *With the above notation, we have  $(v_i) \subseteq \ker(\text{res}_{B, A_i}) \subseteq \sqrt{(v_i)}$ . Slightly more generally, for  $i \neq j$  we have*

$$(v_i) + (v_j) \subseteq \ker(\text{res}_{B, A_i \cap A_j}) \subseteq \sqrt{(v_i) + (v_j)}.$$

*Proof.* The statement about  $\ker(\text{res}_{B, A_i})$  is the content of [15, Proposition 5.2], or equivalently Lemma 3.2. It immediately follows that  $(v_i) + (v_j)$  is contained in  $\ker(\text{res}_{B, A_i \cap A_j})$ . Suppose  $x \in \ker(\text{res}_{B, A_i \cap A_j})$ . Then

$$\text{res}_{B, A_i}(x) \in \ker(\text{res}_{A_i, A_i \cap A_j}).$$

If we can show that  $\ker(\text{res}_{A_i, A_i \cap A_j}) \subseteq \sqrt{(\text{res}_{B, A_i}(v_j))}$ , we are done. We invoke Lemma 4.2: since  $A_i/(A_i \cap A_j) \cong C[u_j]$ , then  $\ker(\text{res}_{A_i, A_i \cap A_j})$  is contained in  $\sqrt{(v)}$ , where  $v$  is the homomorphic image of the polynomial generator in  $H^*(A_i//A_i \cap A_j)$ . This  $v$  is precisely  $\text{res}_{B, A_i}(v_j)$ .  $\blacksquare$



For each  $i$ , we have an element  $z_i \in H^*(A_i)$ ; by the Hopkins–Smith theorem, we can replace each  $z_i$  by  $z_i^{p^n}$  so that there is an element  $x_i \in H^*(B)$  which restricts to  $z_i$ . There are many choices for such an  $x_i$ —by Lemma 4.3, any element in the coset  $x_i + (v_i)$  will do. We want to find a single element  $x$  which restricts to each  $z_i$ . We may raise everything to a large power; then we want to show that for some  $r$ , the set

$$\bigcap_{i=1}^n (x_i^{p^r} + (v_i))$$

is nonempty. We do this by induction: clearly,  $x_1 + (v_1)$  is nonempty. Assume that for some  $q$  we have an element  $y \in \bigcap_{i=1}^{k-1} (x_i^{p^q} + (v_i))$ . For some  $r$ , we want an element in

$$\bigcap_{i=1}^k (x_i^{p^r} + (v_i)) = \left( \bigcap_{i=1}^{k-1} (y^{p^{r-q}} + (v_i)) \right) \cap (x_k^{p^r} + (v_k)),$$

so it certainly suffices to find an element in

$$(y^{p^{r-q}} + (v_1 v_2 \cdots v_{k-1})) \cap (x_k^{p^r} + (v_k)).$$

Since  $y^{p^{r-q}}$  and  $x_k^{p^r}$  restrict compatibly to  $A_i \cap A_k$ , then  $y^{p^{r-q}} - x_k^{p^r} \in \ker(\text{res}_{B, A_i \cap A_k})$  for each  $i$ . By Lemma 4.3, if we choose  $r$  large enough, then for each  $i$  we have elements  $a_i$  and  $b_i$  so that

$$y^{p^{r-q}} - x_k^{p^r} - b_i v_k = a_i v_i. \quad (2)$$

Now we multiply the equations (2),  $i = 1, \dots, k-1$ , together (the first with some multiplicity so that the number of factors is a power of  $p$ , say,  $p^s$ ) to get

$$y^{p^{r+s-q}} - x_k^{p^{r+s}} + c v_k = d v_1 v_2 \cdots v_{k-1}$$

for some elements  $c$  and  $d$ . This equality shows that the desired intersection is nonempty, and so completes the induction. ■

## APPENDIX A: SOME QUASI-ELEMENTARY HOPF ALGEBRAS

The above definitions of Serre elements and quasi-elementary Hopf algebras are a bit on the abstract side. In this appendix, we recall from [15, Proposition 5.1] an alternate characterization of Serre elements of positive internal degree, we give a different description of connected quasi-elementary Hopf algebras, and we give a few examples.

LEMMA A.1 [15]. *Let  $B$  be a graded cocommutative Hopf algebra over a field  $k$  of characteristic  $p$ . Fix a nonzero  $v \in \text{Ext}_B^{2,n}(k, k)$ , with  $n > 0$ .*

(a) *Suppose  $p = 2$ . Then  $v$  is a Serre element if and only if  $v = w^2$  for some  $w \in \text{Ext}_B^1(k, k) \cap \ker \widetilde{\text{Sq}}^0$ .*

(b) *Suppose  $p$  is odd. Then  $v$  is a Serre element if and only if  $v = w^2$  for some  $w \in \text{Ext}_B^{1,m}(k, k)$  with  $m$  odd, or  $v = \beta \tilde{\mathcal{P}}^0(w)$  for some  $w \in \text{Ext}_B^{1,m}(k, k) \cap \ker \tilde{\mathcal{P}}^0$  with  $m$  even.*

(Wilkerson proves, for example, that if  $w \in \ker \widetilde{\text{Sq}}^0$ , then  $w^2$  is a Serre element; the converse is immediate.) The following seems like a reasonable compromise between the above case and the group algebra case (in which  $\widetilde{\text{Sq}}^0$  acts as the identity), and is inspired by the possible actions of  $\widetilde{\text{Sq}}^0$  (or  $\tilde{\mathcal{P}}^0$ ) on  $\text{Ext}_{C(x)}^1(k, k)$ . We do not have enough evidence to grace this with the label of “Conjecture.”

GUESS A.2. *Fix  $v \in \text{Ext}_B^{1,0}(k, k)$ .*

(a) *For  $p = 2$ ,  $v$  is a Serre element if and only if  $v = w^2$  for some eigenvector  $w$  of  $\widetilde{\text{Sq}}^0$ .*

(b) *For  $p$  odd,  $v$  is a Serre element if and only if  $v = \beta \tilde{\mathcal{P}}^0(w)$  for some eigenvector  $w$  of  $\tilde{\mathcal{P}}^0$ .*

We can use Lemma A.1 to give a slightly different characterization of quasi-elementary Hopf algebras, at least in the connected case.

PROPOSITION A.3. *Suppose  $E$  is a finite-dimensional graded connected cocommutative Hopf algebra. Then  $E$  is quasi-elementary if and only if no product of the form*

$$\begin{aligned} \prod_{w \in S} w, & \quad p = 2, \\ \left( \prod_{u \in S_{\text{odd}}} u \right) \left( \prod_{v \in S_{\text{even}}} \beta \tilde{\mathcal{P}}^0 v \right), & \quad p \text{ odd,} \end{aligned}$$

*is nilpotent, where  $S$  is a subset of  $\text{Ext}_E^1(k, k) - \{0\}$ ,  $S_{\text{odd}}$  is a subset of  $\text{Ext}_E^{1,\text{odd}}(k, k) - \{0\}$ , and  $S_{\text{even}}$  is a subset of  $\text{Ext}_E^{1,\text{even}}(k, k) - \{0\}$ .*

*Proof.* We prove this for  $p = 2$ , leaving the other case for the reader. Certainly if no product  $\prod_{w \in S} w$  is nilpotent, then  $E$  is quasi-elementary. So assume that  $E$  is quasi-elementary, and consider  $\prod_{w \in S} w$  where  $S \subseteq \text{Ext}_E^1(k, k) - \{0\}$ . We may assume that  $S$  is of the form  $S = \{w_{ij} : 1 \leq i \leq m, 1 \leq j \leq n_i\}$ , where for each  $i \leq m$ ,  $j < n_i$ , we have  $\widetilde{\text{Sq}}^0 w_{ij} = w_{i,j+1}$ , and  $\widetilde{\text{Sq}}^0 w_{i,n_i} = 0$ . We want to show that  $\prod_{w \in S} w$  is nonnilpotent; we do this by induction on  $N = \sum (n_i - 1)$ , the number of  $w_{ij}$ 's not in the kernel of  $\widetilde{\text{Sq}}^0$ . If  $N$  is zero, then by Lemma A.1,  $\prod_{w \in S} w$  is nonnilpotent. Assume

we have shown that every such product with at most  $N - 1$  factors not in  $\ker(\widetilde{\text{Sq}}^0)$  is nonnilpotent, and suppose  $S$  has  $N$  factors in  $\ker(\widetilde{\text{Sq}}^0)$ ,  $N + m$  factors altogether. Then

$$\widetilde{\text{Sq}}^m \left( \prod w_{ij} \right) = \prod_{\substack{i \leq m \\ j < n_i}} \widetilde{\text{Sq}}^0(w_{ij}) \prod_{i \leq m} \widetilde{\text{Sq}}^1(w_{i, n_i}) = \prod_{\substack{i \leq m \\ j < n_i}} w_{i, j+1} \prod_{i \leq m} w_{i, n_i}^2$$

has fewer terms in  $\ker \widetilde{\text{Sq}}^0$ . Since this last product must then be nonnilpotent, so must  $\prod w_{ij}$ . ■

We conclude with a number of examples of quasi-elementary Hopf algebras. These are, for the most part, reasonably nice Hopf algebras; however, there is no reason to expect all quasi-elementary Hopf algebras to be so nicely behaved. (See, for example, recent work of Bajer and Carlson.)

EXAMPLE A.4. Suppose  $E$  is a  $p$ -group; then  $kE$  is quasi-elementary if and only if  $E$  is elementary abelian. This is the content of Serre's theorem on products of Bocksteins [14]. We should also point out that if  $G$  is a  $p$ -group, then every  $w \in \text{Ext}_{kG}^1(k, k)$  gives rise to a Serre element (either  $w^2$  or  $\beta(w)$ , depending on whether  $p$  is 2 or odd).

EXAMPLE A.5. Suppose  $E$  is the universal enveloping algebra of a restricted Lie algebra. It is possible for  $E$  to be quasi-elementary, but not elementary. See [15, Example 6.5].

EXAMPLE A.6. Suppose  $E$  is a finite-dimensional sub-Hopf algebra of the mod 2 Steenrod algebra. Then  $E$  is quasi-elementary if and only if  $E$  is elementary (see [15, Theorem 6.4]). In terms of the dual of  $A$ , this means that for some  $m \geq 1$ ,  $E$  is a sub-Hopf algebra of

$$(\mathbf{F}_2[\xi_m, \xi_{m+1}, \xi_{m+2}, \dots] / (\xi_m^{2^m}, \xi_{m+1}^{2^m}, \xi_{m+2}^{2^m}, \dots))^*.$$

EXAMPLE A.7. Let  $p$  be an odd prime, and suppose  $E$  is a finite-dimensional sub-Hopf algebra of the mod  $p$  reduced powers. Then  $E$  is quasi-elementary if and only if, for some  $m \geq 2$ ,  $E$  is a sub-Hopf algebra of

$$(\mathbf{F}_p[\xi_{m-1}, \xi_m, \xi_{m+1}, \xi_{m+2}, \dots] / (\xi_{m-1}^{p^m}, \xi_m^{p^m}, \xi_{m+1}^{p^m}, \xi_{m+2}^{p^m}, \dots))^*.$$

See [9, 10, 15].

EXAMPLE A.8. Let  $p$  be an odd prime, and suppose  $E$  is a finite-dimensional sub-Hopf algebra of the mod  $p$  Steenrod algebra. Then  $E$  is

quasi-elementary if and only if  $E$  is a sub-Hopf algebra of one of the following:

$$\begin{aligned} & (E[\tau_0, \tau_1, \tau_2, \dots])^*, \\ & (E[\tau_1, \tau_2, \tau_3, \dots] \otimes \mathbf{F}_p[\xi_1, \xi_2, \xi_3, \dots] / (\xi_1^p, \xi_2^p, \xi_3^p, \dots))^*, \\ & (E[\tau_m, \tau_{m+1}, \dots] \otimes \mathbf{F}_p[\xi_{m-1}, \xi_m, \dots] / (\xi_{m-1}^p, \xi_m^p, \xi_{m+1}^p, \xi_{m+2}^p, \dots))^*, \\ & m \geq 2. \end{aligned}$$

See [9].

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## REFERENCES

1. J. F. Adams and H. R. Margolis, Modules over the Steenrod algebra, *Topology* **10** (1971), 271–282.
2. D. J. Benson, “Representations and Cohomology,” Vols. I and II, Cambridge Univ. Press, Cambridge, UK, 1991.
3. J. F. Carlson, Complexity and Krull dimension, in “Representations of Algebras” (M. Auslander and E. Lluís, Eds.), pp. 62–67, Springer Lecture Notes in Mathematics, Vol. 903, Springer-Verlag, Berlin/New York.
4. L. G. Chouinard, Projectivity and relative projectivity over group rings, *J. Pure Appl. Algebra* **7** (1976), 287–302.
5. E. S. Devinatz, M. J. Hopkins, and J. H. Smith, Nilpotence and stable homotopy theory, I, *Ann. Math. (2)* **128** (1988), 207–241.
6. M. J. Hopkins and J. H. Smith, Nilpotence and stable homotopy theory, II, preprint.
7. O. Kroll, Complexity and elementary abelian  $p$ -groups, *J. Algebra* **88** (1984), 155–172.
8. J. P. May, A general algebraic approach to Steenrod operations, in “The Steenrod Algebra and Its Applications” (F. P. Peterson, Ed.), pp. 153–231, Springer Lecture Notes in Mathematics, Vol. 168, Springer-Verlag, Berlin/New York.
9. D. K. Nakano and J. H. Palmieri, Support varieties for the Steenrod algebra, *Math. Z.*, in press.
10. J. H. Palmieri, Nilpotence for modules over the mod 2 Steenrod algebra, I, *Duke Math. J.* **82** (1996), 195–208.

11. B. Pareigis, On the cohomology of modules over Hopf algebras, *J. Algebra* **22** (1972), 161–182.
12. D. G. Quillen, The spectrum of an equivariant cohomology ring, I, II, *Ann. of Math. (2)* **94** (1971), 549–572, 573–602.
13. D. Radford, Pointed Hopf algebras are free over Hopf subalgebras, *J. Algebra* **45** (1977), 266–273.
14. J.-P. Serre, Sur la dimension cohomologique des groupes profinis, *Topology* **3** (1965), 413–420.
15. C. Wilkerson, The cohomology algebras of finite dimensional Hopf algebras, *Trans. Amer. Math. Soc.* **264** (1981), 137–150.